ELSEVIER
Topology and its Applications 99 (1999) 111–116
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

On spaces that are l-equivalent to a disk

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Received 10 October 1997; received in revised form 6 March 1998

Abstract

For a Tychonoff space X, $C_p(X)$ denotes the space of all real-valued continuous functions on X with the pointwise convergent topology. Two Tychonoff spaces X and Y are said to be l-equivalent if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. We give a characterization of spaces that are l-equivalent to a disk. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Function space; Pointwise convergent topology; Disk

AMS classification: 46E10; 54C35

1. Introduction and preliminaries

In this paper, we assume that all spaces under consideration are Tychonoff. Let $C_p(X)$ be the space of all real-valued continuous functions on X with the pointwise convergent topology, and $C_p(X|A)$ the subspace of $C_p(X)$ consisting of all functions vanishing on a closed subset A of X. For linear topological spaces V and W, $V \sim W$ means that V and W are linearly homeomorphic. Two spaces X and Y are said to be *l-equivalent* if $C_p(X) \sim C_p(Y)$, and written $X \sim_l Y$. By the symbols $\mathbb{D}^n$ and S we specify the n-disk and the convergent sequence, respectively. A space X is said to be *S-stable* if $X \times S \sim_l X$. A compact metric space is called a *compactum*. For a subset Y of X and a subset of Z of Y, we set:

$$\mathrm{Int}_Y Z = \text{the interior of } Z,$$

$$\mathrm{Fr}_Y Z = \text{the boundary of } Z$$

in the subspace Y, respectively; if $Y = X$, we write $\mathrm{Int}\, Z$ for $\mathrm{Int}_X Z$. Other undefined terms can be found in [2–4].

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0166-8641/99/$ – see front matter
PII: S0166-8641(98)00080-7

In 1980, Pavlovskiĭ [7] proved that every n-dimensional finite polyhedron is l-equivalent to the n-disk. Subsequently, several general topologists investigated spaces that are l-equivalent to the n-disk (for example, see [1,5,6,9]). In particular, Arhangel'skiĭ [1] introduced a powerful notation "S-stability" and subsequently obtained an interesting progress. In [5], Kawamura and the author gave the generalization of Pavlovskiĭ's result to topological manifolds. On the other hand, Valov [9] applied Pełczyński's method to an investigation of the C_p-theory and obtained characterizations of spaces that are l-equivalent to the universal Menger compactum and to the Hilbert cube. The purpose of this paper is to give an inner characterization of spaces that are l-equivalent to the n-disk.

2. The main theorem

Definition 2.1. Let X be a finite-dimensional compactum. We put:

$$\mathrm{DK}(X) = \{x \in X \mid \operatorname{ind}_x X = \dim X\}, \quad \text{and}$$
$$\mathrm{M}(X) = \text{the closure of } \mathrm{DK}(X) \text{ in } X.$$

For the definition of $\operatorname{ind}_x X$, see [4, Problem 1.1.B, p. 7]. The subset $\mathrm{DK}(X)$ of X is a known one as the *dimensional kernel* of X.

Definition 2.2. An n-dimensional space X is said to be an $\mathrm{MU}(n)$-*space* if every subset Y of X with $\dim Y = n$ satisfies $\operatorname{Int} Y \neq \emptyset$ and contains a copy of $\mathbb{D}^n$.

Euclidean n-space is an $\mathrm{MU}(n)$-space; this classical result that is due to Menger and Urysohn is well known. By using the above two definitions, we can now formulate a theorem that gives a characterization.

Theorem 2.3. *For a space X and a natural number n, X is l-equivalent to $\mathbb{D}^n$ if and only if X is an n-dimensional compactum with*:

(2.3.1) *there exists a non-empty open subset of $\mathrm{M}(X)$ that is an $\mathrm{MU}(n)$-space,*

(2.3.2) *every non-empty open subset of $\mathrm{M}(X)$ contains a subset that is l-equivalent to X.*

To prove the theorem, we need some lemmas.

The following lemma is due to Menger (see [4, Problem 1.5.H.(c), p. 38]).

Lemma 2.4 (Menger). *Let X be an n-dimensional compactum. For every point x in $\mathrm{DK}(X)$, $\operatorname{ind}_x \mathrm{DK}(X) = n$.*

By Lemma 2.4, we have:

Lemma 2.5. *Let X be an n-dimensional compactum. If U is a non-empty open subset of $\mathrm{M}(X)$, then $\dim U = n$.*

As Lemma 2.5 may be known, for the completeness of the paper, we give a proof.

Proof. Suppose that $\dim U \leqslant n-1$ and take a point x in $U \cap \mathrm{DK}(X)$. Since $\mathrm{ind}_x \mathrm{DK}(X) = n$ by Lemma 2.4, there exists a neighbourhood V of x in $\mathrm{DK}(X)$ such that, for every neighbourhood W of x in $\mathrm{DK}(X)$ with $W \subset V$, $\mathrm{ind}\,\mathrm{Fr}_{\mathrm{DK}(X)} W > n-2$. Take a neighbourhood N of x in $\mathrm{M}(X)$ such that $N \cap \mathrm{DK}(X) = V$. By our assumption, there exists an open subset G of U with $x \in G \subset N \cap U$ and $\mathrm{ind}\,\mathrm{Fr}_U G \leqslant n-2$. On the other hand, obviously, we have $\mathrm{Fr}_{\mathrm{DK}(X)}(G \cap \mathrm{DK}(X)) \subset \mathrm{Fr}_U G$. This is a contradiction. □

Lemma 2.6 [7]. *For two compacta X and Y with $X \sim_l Y$*:

(1) *if Z is a subset of X with the Baire property, then there exists a non-empty open subset of Z that is homeomorphic to a subset of Y*,

(2) $\dim X = \dim Y$.

Lemma 2.7 [1, Proposition 19]. *If two compacta X and Y are l-equivalent and Z is an arbitrary space, then $X \times Z \sim_l Y \times Z$.*

Lemma 2.8 [1, Proposition 20]. *For any space X, $X \times S$ is S-stable.*

Lemma 2.9 [1, Proposition 22]. *Let X be a compactum. If X contains a closed subset A such that $A \sim_l X \times S$, then X is S-stable.*

Lemma 2.9 easily induces:

Lemma 2.10. *The n-disk $\mathbb{D}^n$ is S-stable.*

Lemma 2.11. *If a space X is S-stable, then*:

(1) $C_p(X) \times C_p(X) \sim C_p(X)$,

(2) *for every space Y with $Y \sim_l X$, Y is S-stable.*

Proof. For (1)

$$C_p(X) \times C_p(X) \sim C_p(X \times S) \times C_p(X) \sim C_p\big((X \times S) \oplus X\big) \sim C_p(X \times S) \sim C_p(X).$$

Lemma 2.7 implies (2) as follows:

$$Y \times S \sim_l X \times S \sim_l X \sim_l Y. \qquad \square$$

The following lemma is well known (for example, see [9, p. 585]).

Lemma 2.12. *Let X be a compactum. If A is a closed subset of X, then $C_p(X) \sim C_p(X|A) \times C_p(A)$.*

Lemma 2.13. *If a space X is l-equivalent to some compactum, then X is a compactum.*

Proof. Since the network weight and compactness are preserved by l-equivalence (see [2, Theorem I.1.3, p. 26] and [8]), X is a compactum. □

Proof of the main theorem. ('if') Suppose that an n-dimensional compactum X satisfies the conditions (2.3.1) and (2.3.2). First, we show that $X \times S \sim_l \mathbb{D}^n$. By the condition (2.3.1), there exists a non-empty open subset U of $\mathrm{M}(X)$ that is an $\mathrm{MU}(n)$-space. Furthermore, by the condition (2.3.2), U contains a subset Y_1 with $Y_1 \sim_l X$. Since, by Lemma 2.6(2), $\dim Y_1 = n$, and Y_1 contains a copy A of $\mathbb{D}^n$. Then $\mathrm{Int}_U A \neq \emptyset$ because $\dim A = n$. Applying the condition (2.3.2) again, we can find a subset Y_2 of $\mathrm{Int}_U A$ such that $Y_2 \sim_l X$. Then, $Y_2 \times S \subset A \times S \subset Y_1 \times S$, and, by Lemma 2.7, we have

$$Y_1 \times S \sim_l X \times S \sim_l Y_2 \times S.$$

It follows that

$$\begin{aligned}
&C_p(X \times S) \sim C_p(Y_1 \times S) \\
&\quad \sim C_p(Y_1 \times S \mid A \times S) \times C_p(A \times S) \text{ (by Lemma 2.12)} \\
&\quad \sim C_p(Y_1 \times S \mid A \times S) \times C_p(A \times S) \times C_p(A \times S) \text{ (by Lemmas 2.8 and 2.11(1))} \\
&\quad \sim C_p(Y_1 \times S) \times C_p(A \times S) \text{ (by Lemma 2.12)} \\
&\quad \sim C_p(Y_2 \times S) \times C_p(A \times S) \\
&\quad \sim C_p(Y_2 \times S) \times C_p(Y_2 \times S) \times C_p(A \times S \mid Y_2 \times S) \text{ (by Lemma 2.12)} \\
&\quad \sim C_p(Y_2 \times S) \times C_p(A \times S \mid Y_2 \times S) \text{ (by Lemmas 2.8 and 2.11(1))} \\
&\quad \sim C_p(A \times S) \text{ (by Lemma 2.12)} \\
&\quad \sim C_p(\mathbb{D}^n \times S) \text{ (by Lemma 2.7)} \\
&\quad \sim C_p(\mathbb{D}^n) \text{ (by Lemma 2.10).}
\end{aligned}$$

Hence $X \times S \sim_l \mathbb{D}^n$. Thereby X is S-stable because of the condition (2.3.1) and Lemma 2.9. It follows that $X \sim_l X \times S \sim_l \mathbb{D}^n$.

('only if') By Lemmas 2.6(2) and 2.13, X is an n-dimensional compactum. Now we proceed to check the conditions (2.3.1) and (2.3.2). By Lemma 2.6(1), there exists a non-empty open subset U of $\mathrm{M}(X)$ that is homeomorphic to a subset of $\mathbb{D}^n$. By Lemma 2.5, $\dim U = n$. It is easy to see that U is an $\mathrm{MU}(n)$-space. To check the condition (2.3.2), take a non-empty open subset V of $\mathrm{M}(X)$. Since $\mathrm{M}(X)$ is closed in X and V is open in $\mathrm{M}(X)$, V has the Baire property. By Lemma 2.6(1) again, there exists a non-empty open subset W of V that is homeomorphic to a subset of $\mathbb{D}^n$. Since $\dim W = n$ by Lemma 2.5, W contains a copy of $\mathbb{D}^n$ that is l-equivalent to X by the assumption. □

3. Corollaries and related matters

Corollary 3.1. *If $X \sim_l \mathbb{D}^n$ then $X \sim_l \mathrm{M}(X)$.*

Proof. By Theorem 2.3, X is a compactum. Moreover, by the condition (2.3.2) and Lemma 2.13, there exists a compact subset Z of $\mathrm{M}(X)$ such that $Z \sim_l X$. Now we have:

$$\begin{aligned}
&C_p\big(\mathrm{M}(X) \times S\big) \\
&\quad \sim C_p\big(\mathrm{M}(X) \times S \mid Z \times S\big) \times C_p(Z \times S) \text{ (by Lemma 2.12)}
\end{aligned}$$

$$\begin{aligned}
&\sim C_p\big(\mathrm{M}(X)\times S \mid Z\times S\big)\times C_p(Z\times S)\times C_p(Z\times S)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Lemmas 2.8 and 2.11(1))}\\
&\sim C_p\big(\mathrm{M}(X)\times S\big)\times C_p(Z\times S)\ \text{(by Lemma 2.12)}\\
&\sim C_p\big(\mathrm{M}(X)\times S\big)\times C_p(X\times S)\ \text{(by Lemma 2.7)}\\
&\sim C_p\big(\mathrm{M}(X)\times S\big)\times C_p\big(\mathrm{M}(X)\times S\big)\times C_p\big(X\times S \mid \mathrm{M}(X)\times S\big)\ \text{(by Lemma 2.12)}\\
&\sim C_p\big(\mathrm{M}(X)\times S\big)\times C_p\big(X\times S \mid \mathrm{M}(X)\times S\big)\ \text{(by Lemmas 2.8 and 2.11(1))}\\
&\sim C_p\big(X\times S\big)\ \text{(by Lemma 2.12)}\\
&\sim C_p(X)\ \text{(by Lemma 2.11(2))}\\
&\sim C_p(Z).
\end{aligned}$$

By Lemma 2.9, we have $X \sim_l \mathrm{M}(X)$. □

In [5], Kawamura and the author proved that every compact topological n-manifold is l-equivalent to $\mathbb{D}^n$. Unfortunately, we cannot set this result as a corollary of Theorem 2.3 because it is difficult to check the condition (2.3.2) directly for several compact topological manifolds. Now, we introduce a class of spaces for that we have no difficulty in checking the condition.

Definition 3.2. A space X is said to be *densely self-embeddable* if every non-empty open subset of X contains a copy of X.

Corollary 3.3. *For a densely self-embeddable space X and a natural number n, X is l-equivalent to $\mathbb{D}^n$ if and only if X is an n-dimensional* MU(n)*-compactum.*

Proof. First, notice that $\mathrm{M}(X) = X$. Suppose that X is an n-dimensional MU(n)-compactum. It is easy to see that X satisfies the conditions (2.3.1) and (2.3.2). Now we proceed to show the converse. By Theorem 2.3, X satisfies the condition (2.3.1). Since X is densely self-embeddable, we may assume that X is a subset of the MU(n)-space U. Notice that every n-dimensional subset of an MU(n)-space is an MU(n)-space. □

Corollary 3.4. *For every densely self-embeddable space X, the space $X\times$* (the Cantor set) *is not l-equivalent to $\mathbb{D}^n$.*

Remarks. It is well known that the n-dimensional universal Menger compactum μ^n is densely self-embeddable. Thus, we have $\mathrm{M}(\mu^1) = \mu^1$. By these facts, we can see that the space μ^1 satisfies the condition (2.3.2), but not (2.3.1) for $n = 1$. Moreover, Valov [9, Theorem 2.9] proved that a space X is l-equivalent to μ^n if and only if X is an n-dimensional compactum containing a copy of μ^n. By this fact, we can see that μ^1 is not l-equivalent to $\mathbb{D}^1$, and thus, the converse of Corollary 3.1 does not hold. Obviously, the space $\mu^1 \oplus \mathbb{D}^1$ satisfies the condition (2.3.1). By the above result of Valov, $\mu^1 \oplus \mathbb{D}^1$ is l-equivalent to μ^1, and thus, it does not satisfy the condition (2.3.2). Thus, we cannot drop each one of the conditions (2.3.1) and (2.3.2) in Theorem 2.3. Next, we refer to the existence of a space X that is not l-equivalent to $\mathrm{M}(X)$ (cf. Corollary 3.1). Let X be the space $\mu^1 \oplus \mathbb{D}^2$. Obviously, we have $\mathrm{M}(X) = \mathbb{D}^2$. Suppose that X is l-equivalent to $\mathbb{D}^2$. By Lemma 2.6(1), there exists a non-empty open subset of μ^1 that is homeomorphic to a subset of $\mathbb{D}^2$. This contradicts the fact that μ^1 is densely self-embeddable. In conclusion, we point out the existence of a space X with $X \sim_l \mathbb{D}^1$ that is not an MU(1)-space (cf. Corollary 3.3). Let X be the space $\mathbb{D}^1 \times S$. By Lemma 2.10, X is l-equivalent to $\mathbb{D}^1$. The interior of the 1-dimensional compact subset $\mathbb{D}^1 \times \{p\}$ of X is empty, where p is the limit point of S. Notice that $\mathrm{M}(X) = X$ for this space X.

References

[1] A.V. Arhangel'skiĭ, On linear topological classification of spaces of continuous functions in the topology of pointwise convergence, Mat. Sb. 70 (1991) 129–142.

[2] A.V. Arhangel'skiĭ, Topological Function Spaces (Kluwer Academic Publishers, Dordrecht, 1992).

[3] R. Engelking, General Topology (Heldermann, Berlin, 1989).

[4] R. Engelking, Theory of Dimensions, Finite and Infinite (Heldermann, Berlin, 1995).

[5] K. Kawamura and K. Morishita, Linear topological classification of certain function spaces on manifolds and CW complexes, Topology Appl. 69 (1996) 265–282.

[6] A. Koyama and T. Okada, On compacta which are l-equivalent to I^n, Tsukuba J. Math. 11 (1987) 147–156.

[7] D.S. Pavlovskiĭ, On spaces of continuous functions, Soviet Math. Dokl. 22 (1980) 34–37.

[8] V.V. Uspenskiĭ, A characterization of compactness in terms of uniform structure in a function space, Russian Math. Surveys 37 (1982) 143–144.

[9] V.M. Valov, Linear topological classifications of certain function spaces, Trans. Amer. Math. Soc. 327 (1991) 583–600.